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# WIENER MULTICHANNEL FILTERING IN THE FREQUENCY DOMAIN

Prepared by

William H. Osborne

TEXAS INSTRUMENTS INCORPORATED
Science Services Division
P.O. Box 5621
Dallas, Texas 75222

Prepared for

EXPLORATORY DEVELOPMENT

NAVAL SHIPS SYSTEMS COMMAND

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## SECTION I
## INTRODUCTION

For the past several years, both the Navy and industry have expended much effort in the area of optimum beamforming. Impetus for this work has come from results in underwater acoustics and signal processing areas. As various underwater acoustics programs developed, it was learned that many of the noise fields encountered by current acoustic systems were neither spatially random nor isotropic, and in many cases, assumptions of spatially random or isotropic noise fields did not even remotely approximate the actual noise environment. As these facts developed, it became clear that optimum beamforming offered potential gains that were too significant to be ignored.

The goal of optimum beamforming is to form a beam in the desired direction while minimizing the interfering noise arriving at the array from other directions. In those cases where there are spatially discrete interfering noise sources, optimum beamforming designs a beam that has nulls directed toward those interfering sources. The depth and directivity of the nulls will determine the amount of gain that is actually achieved against a noise field of this kind. If the noise field has no spatially discrete components, the optimum beamformer gives the same performance as the conventional time-shift-and-sum beamformer. An optimum beamformer never fails to at least match the performance of a conventional beamformer since the optimum beamformer always samples the noise field against which it must process and uses this information in the design of its beam. Thus, many of the arbitrary qualities associated with time-shift-and-sum beamforming are not present in optimum beamforming.

Almost concurrent with the developments in underwater acoustics were significant developments in digital signal processing techniques. One of the most significant developments was the fast Fourier transform (FFT) algorithm. The FFT development prompted the evolution of a whole new series of algorithms for the digital implementation of optimum beamforming.

In the early stages of Navy application, all optimum beamforming was done in the time domain. The time-domain approach was severely limited by the large number of arithmetic operations required to implement the digital filters. However, the advent of the FFT opened new avenues for approaching the problem. With the capability for doing frequency-domain processing, a series of new algorithms were developed. These algorithms were much more reasonable in that they greatly reduced the number of arithmetic operations required to implement digital filters.

This report ~~will~~ presents the theory behind the frequency-domain approach to optimum beamforming as implemented with Wiener multichannel filters, along with algorithms for actually designing the filters. There are other approaches to optimum beamforming that could be taken with comparable success. However, Wiener's approach is the optimum under the least-mean-square criterion for signal extraction. For this reason, Texas Instruments has chosen to use Wiener's approach for initial evaluation of optimum beamforming before considering other approaches that approximate optimum.

Another subject proven valuable in studying the ability of multichannel filters to perform is the frequency-wavenumber spectrum. The frequency-wavenumber spectrum is used to characterize the acoustic noise field that the filters must process against. Since this technique provides information essential to the understanding of optimum beamforming performance, this report will include a discussion of frequency-wavenumber spectra.

Finally, to enable comparison of optimum beamforming with time-shift-and-sum beamforming, array gain techniques will be discussed. Along with the actual array gain calculation, the type of comparisons that can be performed are considered.

## SECTION II

## WIENER MULTICHANNEL FREQUENCY-DOMAIN FILTERING

To provide the necessary background for understanding Wiener filtering and the problems encountered when actually computing the filters, a complete theoretical discussion of Wiener filtering will be undertaken. A derivation will be given which provides the exact solution for the Wiener multichannel filters in the frequency domain. Since digital filtering on a computer requires that filters be limited to a certain total length, it will, in general, be impossible to apply these frequency filters exactly because their inverse Fourier transform ordinarily will be of infinite length in time. The derivation is given because it is needed to obtain an understanding of multichannel filtering in the frequency domain.

The performance criterion in Wiener linear multichannel filtering is to minimize the difference $e(t)$ between the signal $s(t)$ and the output $g(t)$ of a Wiener filter set $\left\{a_j(\tau) \mid j = 1, 2, \ldots, n; -\infty < \tau < \infty\right\}$ by minimizing the mean-square-error I given by

$$I = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[g(t) - s(t)\right]^2 dt$$

where $g(t)$ is the output of the multichannel filter set obtained by convolution filtering of each channel and subsequent summation of the filtered time traces:

$$g(t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} a_j(\tau)\, f_j(t-\tau)\, d\tau$$

From the mean-square-error, we obtain

$$I = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \left[ \sum_{j=1}^{n} \int_{-\infty}^{\infty} a_j(\tau)\, f_j(t-\tau)\, d\tau - s(t) \right]^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_j(\tau)\, f_j(t-\tau)\, a_k(\tau')\, f_k(t-\tau')\, d\tau\, d\tau' \right] dt$$

$$- 2 \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} s(t) \left[ \sum_{j=1}^{n} \int_{-\infty}^{\infty} a_j(\tau)\, f_j(t-\tau)\, d\tau \right] dt$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \left[ s(t) \right]^2 dt$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_j(\tau)\, a_k(\tau') \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_j(t) f_k \left[ t - (\tau' - \tau) \right] dt \right\} d\tau\, d\tau'$$

$$-2 \sum_{j=1}^{n} \int_{-\infty}^{\infty} a_j(\tau) \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\, f_j(t-\tau)\, dt \right\} d\tau$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[ s(t) \right]^2 dt$$

Let $\tau' = x + \tau$. Then, the mean square error I is

$$I = \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_j(\tau)\, a_k(\tau + x) \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_j(t)\, f_k(t-x)dt \right\} d\tau dx$$

$$- 2\sum_{j=1}^{n} \int_{-\infty}^{\infty} a_j(\tau) \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\ f_j(t-\tau)\ dt \right\} d\tau$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[s(t)\right]^2 dt$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} a_j(\tau)\, a_k(\tau + x)\ d\tau \right] \left[ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_j(t)\, f_k(t-x)dt \right] dx$$

$$- 2\sum_{j=1}^{n} \left[ \int_{-\infty}^{\infty} a_j(\tau) \right] \left[ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\ f_j(t-\tau)\ dt \right] d\tau$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[s(t)\right]^2 dt$$

The correlation function $\varphi_{jk}(\tau)$ between the $k^{th}$ channels at a time lag $\tau$ is defined to be

$$\varphi_{jk}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_j(t-\tau)\, f_k(t)\ dt$$

The subscript s will denote correlation with the signal function s(t).

The mean square error I is then simplified to

$$I = \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} a_j(\tau)\, a_k(\tau + x)\, d\tau \right] \varphi_{kj}(x)\, dx$$

$$- 2 \sum_{j=1}^{n} \left[ \int_{-\infty}^{\infty} a_j(\tau)\, \varphi_{js}(\tau)\, d\tau \right] + \varphi_{ss}(0)$$

To see how changing the frequency filter weights

$A_j(f) = \int_{-\infty}^{\infty} a_j(\tau)\, e^{-i2\pi f \tau} d\tau$ affects the value of the mean square error, recall that the Fourier transform and its inverse transform are related in the following manner:

$$\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} a_j(t)\, e^{-i2\pi ft} dt \right] e^{i2\pi f\tau}\, df = a_j(\tau)$$

Hence, an alternative form for the impulse response $a_j(\tau)$ of the frequency filter $A_j(f)$ is given by

$$a_j(\tau) = \int_{-\infty}^{\infty} A_j(f)\, e^{i2\pi f\tau}\, df$$

Recall also that convolution in the time domain or the frequency domain results in multiplication in the frequency domain or the time domain when a Fourier transform is performed. Here, the following result will be used;

$$\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} a_j(\tau)\, a_k(\tau - x)\, d\tau \right] e^{-i2\pi fx} dx = A_j(f)\, A_k(f)$$

so that

$$\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} a_j(\tau)\, a_k(\tau + x)\, d\tau \right] e^{+i2\pi fx} dx = A_j(f)\, A_k(-f) = A_j(f)\, A_k^*(f)$$

where the asterisk denotes complex conjugate.

From the preceding results,

$$I = \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} A_j(f)\, A_k^*(f)\, e^{-i2\pi fx} df \right] \varphi_{kj}(x)\, dx$$

$$- 2 \sum_{j=1}^{n} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} A_j(f)\, e^{i2\pi f\tau} df \right] \varphi_{js}(\tau)\, d\tau + \varphi_{ss}(0)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} A_j(f)\, A_k^*(f) \left[ \int_{-\infty}^{\infty} \varphi_{kj}(x)\, e^{-i2\pi fx} dx \right] df$$

$$- 2 \sum_{j=1}^{n} \int_{-\infty}^{\infty} A_j(f) \left[ \int_{-\infty}^{\infty} \varphi_{js}(\tau)\, e^{i2\pi f\tau} d\tau \right] df + \varphi_{ss}(0)$$

$$= \int_{-\infty}^{\infty} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} A_j(f)\, A_k^*(f)\, \Phi_{jk}^*(f) - 2 \sum_{j=1}^{n} A_j(f)\, \Phi_{sj}(f) \right] df$$

$$+ \varphi_{ss}(0)$$

Since

$$\int_{-\infty}^{\infty} A_j(f)\, \Phi_{sj}(f)\, df = \int_{-\infty}^{\infty} A_j(-f)\, \Phi_{sj}(-f)\, df = \int_{-\infty}^{\infty} A_j^*(f)\, \Phi_{sj}^*(f)\, df$$

it follows that

$$I = \int_{-\infty}^{\infty} \left\{ \sum_{j=1}^{n} \sum_{k=1}^{n} A_j(f)\, A_k^*(f)\, \Phi_{jk}^*(f) - \sum_{j=1}^{n} \left[ A_j(f)\, \Phi_{sj}(f) + A_j^*(f)\, \Phi_{sj}^*(f) \right] \right\} df$$

$$+ \varphi_{ss}(0)$$

Thus, the quantity inside the braces is seen to be a real function of frequency.

To find the filters $A_j(f)$ which minimize the mean-square-error, it is necessary to set in the partial derivatives of the integrand with respect to the real and imaginary parts of $A_j(f) = 0$ for each channel:

$$\sum_{k=1}^{n} A_k(f)\,\Phi_{jk}(f) + A_k^*(f)\,\Phi_{jk}(f) - \Phi_{sj}(f) - \Phi_{sj}^*(f) = 0 ;$$

$$i\left[\sum_{k=1}^{n} A_k^*(f)\,\Phi_{jk}^*(f) - A_k(f)\,\Phi_{jk}(f) - \Phi_{sj}(f) + \Phi_{sj}^*(f)\right] = 0$$

Hence, the optimum filters $A_j(f)$ must be such that

$$\sum_{k=1}^{n} A_k(f)\,\Phi_{jk}(f) = \Phi_{js}(f)$$

Therefore, it is necessary to solve the matrix equation

$$\begin{bmatrix} \Phi_{11}(f) & \Phi_{12}(f) & \cdots & \Phi_{1n}(f) \\ \Phi_{21}(f) & \Phi_{22}(f) & \cdots & \Phi_{2n}(f) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Phi_{n1}(f) & \Phi_{n2}(f) & \cdots & \Phi_{nn}(f) \end{bmatrix} \begin{bmatrix} A_1(f) \\ A_2(f) \\ \cdot \\ \cdot \\ A_n(f) \end{bmatrix} = \begin{bmatrix} \Phi_{1s}(f) \\ \Phi_{2s}(f) \\ \cdot \\ \cdot \\ \Phi_{ns}(f) \end{bmatrix}$$

to obtain the Wiener optimum multichannel frequency filter set $\{A_j(f)\}$.

This matrix equation forms a system of only n equations in the unknowns $A_j(f)$ but must be solved for all of the frequencies f in the band of interest.

Assume that a different frequency filter set $A_j(f) + \Delta_j(f)$ is applied to the particular set of sensor outputs for which the correlation transforms were computed. The mean-square-error I will then be given by

$$I = \int_{-\infty}^{\infty} \Bigg\{ \Phi_{ss}(f) - \sum_{j=1}^{n} \left[ A_j(f) \; \Phi_{sj}(f) + A_j^*(f) \; \Phi_{sj}^*(f) \right]$$

$$- \sum_{j=1}^{n} \left[ \Delta_j(f) \; \Phi_{sj}(f) + \Delta_j^*(f) \; \Phi_{sj}^*(f) \right]$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \left[ A_j(f) + \Delta_j(f) \right] \left[ A_k^*(f) + \Delta_k^*(f) \right] \; \Phi_{jk}^*(f) \Bigg\} \, df$$

$$= \int_{-\infty}^{\infty} \Bigg\{ \Phi_{ss}(f) - \sum_{j=1}^{n} \left[ A_j(f) \; \Phi_{sj}(f) + A_j^*(f) \; \Phi_{sj}^*(f) \right]$$

$$- \sum_{j=1}^{n} \left[ \Delta_j(f) \; \Phi_{sj}(f) + \Delta_j^*(f) \; \Phi_{sj}^*(f) \right]$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \Big[ A_j(f) \, A_k^*(f) \; \Phi_{jk}^*(f) + A_j(f) \; \Delta_k^*(f) \; \Phi_{jk}^*(f)$$

$$+ A_j^*(f) \; \Delta_k(f) \; \Phi_{jk}(f) + \Delta_j(f) \; \Delta_k^*(f) \; \Phi_{jk}^*(f) \Big] \Bigg\} \, df$$

Since

$$\sum_{k=1}^{n} A_k(f)\ \Phi_{jk}(f) = \Phi_{js}(f)$$

and

$$\sum_{k=1}^{n} A_k^*(f)\ \Phi_{jk}^*(f) = \Phi_{js}^*(f)$$

$$\sum_{j=1}^{n}\ \sum_{k=1}^{n} \Delta_j^*(f)\ A_k(f)\ \Phi_{jk}(f) = \sum_{j=1}^{n} \Delta_j^*(f)\ \Phi_{sj}^*(f)$$

and

$$\sum_{j=1}^{n}\ \sum_{k=1}^{n} A_j(f)\ A_k^*(f)\ \Phi_{jk}^*(f) = \sum_{j=1}^{n} A_j(f)\ \Phi_{sj}(f)$$

and

$$\sum_{j=1}^{n}\ \sum_{k=1}^{n} \Delta_j(f)\ A_k^*(f)\ \Phi_{jk}^*(f) = \sum_{j=1}^{n} \Delta_j(f)\ \Phi_{sj}(f)$$

the mean-square-error I is given by

$$I = \int_{-\infty}^{\infty} \left[ \Phi_{ss}(f) - \sum_{j=1}^{n} A_j^*(f)\ \Phi_{sj}^*(f) + \sum_{j=1}^{n}\ \sum_{k=1}^{n} \Delta_j(f)\ \Delta_k^*(f)\ \Phi_{jk}^*(f) \right] df$$

$$= \int_{-\infty}^{\infty} \left[ \Phi_{ss}(f) - \sum_{j=1}^{n} A_j^*(f)\ \Phi_{sj}^*(f) \right] df$$

$$+ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ \sum_{j=1}^{n} \int_{-\infty}^{\infty} \delta_j(\tau)\ f_j(t-\tau)\, d\tau \right]^2 dt$$

Since the bottom expression is positive, the mean-square-error I is a minimum if and only if $\delta_j(\tau) = 0$ for all values of j and $\tau$. This condition is equivalent to the condition that $\Delta_j(f) = 0$ for all values of j and $\tau$. Therefore, the Wiener optimum frequency filter set $\left\{A_j(f)\right\}$ is achieved by satisfying the set of equations

$$\sum_{k=1}^{n} A_k(f)\, \Phi_{jk}(f) = \Phi_{js}(f)$$

Then the mean-square-error I is given by

$$I = \int_{-\infty}^{\infty} \left[ \Phi_{ss}(f) - \sum_{j=1}^{n} A_j^*(f)\, \Phi_{sj}^*(f) \right] df$$

or

$$I = \int_{-\infty}^{\infty} \left[ \Phi_{ss}(f) - \sum_{j=1}^{n} A_j(f)\, \Phi'_{sj}(f) \right] df$$

or

$$I = \int_{-\infty}^{\infty} \left\{ \Phi_{ss}(f) - \frac{1}{2} \left[ \sum_{j=1}^{n} A_j(f)\, \Phi'_{sj}(f) + A_j^*(f)\, \Phi_{sj}^*(f) \right] \right\} df$$

Since

$$A_j(f) = \sum_{k=1}^{n} \left[ \Phi_{jk}(f) \right]^{-1} \Phi_{ks}(f)$$

and

$$A_j^*(f) = \sum \left[ \Phi_{jk}^*(f) \right]^{-1} \Phi_{ks}^*(f)$$

then,

$$\sum_{j=1}^{n} A_j(f)\,\Phi_{sj}(f) = \sum_{j=1}^{n}\sum_{k=1}^{n}\left[\Phi_{jk}(f)\right]^{-1}\Phi_{ks}(f)\,\Phi_{sj}(f)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\left[\Phi_{jk}(f)\right]^{-1}\Phi^*_{js}(f)\,\Phi^*_{sk}(f)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\left[\Phi_{kj}(f)\right]^{-1}\Phi^*_{ks}(f)\,\Phi^*_{sj}(f)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\left[\Phi^*_{jk}(f)\right]^{-1}\Phi^*_{ks}(f)\,\Phi^*_{sj}(f)$$

$$= \sum_{j=1}^{n} A^*_j(f)\,\Phi^*_{sj}(f)$$

The quantity

$$\sum_{j=1}^{n} A_j(f)\,\Phi_{sj}(f)$$

is equal to its complex conjugate and is, therefore, real. It is then reasonable to define a mean-square-error density I(f) as follows:

$$I(f) = \Phi_{ss}(f) - \sum_{j=1}^{n}\sum_{k=1}^{n}\Phi_{sj}(f)\left[\Phi_{jk}(f)\right]^{-1}\Phi_{ks}(f)$$

$$= \Phi_{ss}(f)\left\{1 - \frac{\sum_{j=1}^{n}\sum_{k=1}^{n}\Phi_{sj}(f)\left[\Phi_{jk}(f)\right]^{-1}\Phi_{ks}(f)}{\Phi_{ss}(f)}\right\}$$

The quantity

$$\frac{\sum_{j=1}^{n} \sum_{k=1}^{n} \Phi_{sj}(f) \left[\Phi_{jk}(f)\right]^{-1} \Phi_{ks}(f)}{\Phi_{ss}(f)}$$

is known as the multichannel squared coherence of the signal s(t) with the sensor output set $\left[f_j(t) \mid j = 1, 2, \ldots, n\right]$ at the frequency f. It is a measure of the relative success of a Wiener optimum multichannel filter set $\left[A_j(f)\right]$ in extracting a signal s(t) buried in noise. If it is equal to 1, the mean-square-error density I(f) at the frequency f is 0. If it is 0, the mean-square-error density I(f) is $\Phi_{ss}(f)$, the total signal power density. In general, it lies somewhere between these two extremes so that the relative error $I(f)/\Phi_{ss}(f)$ is equal to 1 minus the multichannel squared coherence.

# SECTION III

# NUMERICAL IMPLEMENTATION

The analysis presented in Section II is an analysis of continuous functions. However, it is not possible to use continuous functions when working with real data. In this case, discrete values are used to represent the continuous functions and numerical methods are used to approximate operations such as integration, etc. This section will be devoted to the numerical calculation of the crosspower matrix and the solution of the matrix equation to obtain the Wiener filters.

## A. FAST FOURIER TRANSFORM (FFT)

Due to the savings in computation time, all processing is done in the frequency domain. Therefore, a form of the FFT is used to transform all time data to the frequency domain.

Suppose a time series having $N = 2^n$ samples is divided into two series, each of which has only half as many points. Denote the original series by $X = (X_0, X_1, X_2, X_3, \ldots, X_n)$ and the other two by $Y = (Y_0, Y_1, \ldots, Y_{N/2})$ and $Z = (Z_0, Z_1, \ldots Z_{N/2})$, respectively. The two smaller series are related to the original by

$$
\begin{aligned}
Y_k &= X_{2k} \\
& \qquad\qquad k = 0, 1, 2, \ldots, (N/2) - 1 \\
Z_k &= X_{2k+1}
\end{aligned}
$$

If the discrete Fourier transform of the time series $X_0$, $X_1$, $\ldots$, $X_{n-1}$ is defined by

$$
A_r = \sum_{k=0}^{N=1} X_k \, e^{-2\pi jrk/N} \qquad r = 0, 1, \ldots, N-1
$$

where $A_r$ is the $r^{th}$ coefficient, $X_k$ is the $k^{th}$ time sample and $j = \sqrt{-1}$, then the discrete transforms of Y and Z are given by

$$B_r = \sum_{k=0}^{(N/2)-1} Y_k \, e^{-4\pi jrk/N} \qquad r = 0, 1, 2, \ldots, (N/2) - 1$$

$$C_r = \sum_{k=0}^{(N/2)-1} Z_k \, e^{-4\pi jrk/N} \qquad r = 0, 1, 2, \ldots, (N/2) - 1$$

The discrete transform of X is then given in terms of odd and even numbered points as

$$A_r = \sum_{k=0}^{(N/2)-1} \left\{ Y_k \, e^{-4\pi jrk/N} + Z_k \, e^{-2\pi jr(2k+1)/N} \right\} \qquad r = 0, 1, \ldots, N-1$$

or

$$A_r = \sum_{k=0}^{(N/2)-1} Y_k \, e^{-4\pi jrk/N} + e^{-2\pi jr/N} \sum_{k=0}^{(N/2)-1} Z_k \, e^{-4\pi jrk/N}$$

$$= B_r + e^{-2\pi jr/N} C_r, \qquad 0 \le r < N/2$$

For values of r greater than N/2, the discrete Fourier transform for $B_r$ and $C_r$ repeat periodically the values taken on when $r < N/2$. Therefore, substituting $r + (N/2)$ for r in the above equation for $A_r$ gives

$$A_{r+(N/2)} = B_r + e^{-2\pi j\,[r+(N/2)]\,/N} C_r$$

$$= B_r - e^{-2\pi jr/N} C_r \qquad 0 \le r < N/2$$

Thus,

$$A_r = B_r + e^{-2\pi jr/N} C_r, \qquad 0 \le r < N/2$$

$$A_{r+N/2} = B_r - e^{-2\pi jr/N} C_r, \qquad 0 \le r < N/2$$

The above process illustrates the reduction that can be achieved. By dividing the sequence X down until the two-point transform is reached and then using the above relations, the Fourier transform of X can be achieved in approximately $1/2\ N \cdot \log_2 N$ complex multiplications and $N \cdot \log_2 N$ additions. The above process results in a scrambled Fourier transform, making one additional step necessary to get the output in the desired format.

## B. CROSSPOWER MATRIX

The crosspower matrix is the set of all cross- and auto-power spectral densities for the various channels. Thus, it is necessary to compute each of the spectral densities to form the matrix. The method outlined by Richard A. Haubrich is a computational procedure that is particularly efficient for computing power spectra.* This method is outlined below.

Let $X_j(n\Delta t)$ be the $n^{th}$ sample on channel j, where $n = 1, 2, \ldots, N$. Subdivide $X_j$ into p subsets and denote the $k^{th}$ subset by $X_{jk}(m\Delta t)$, where $m = 1, 2, \ldots, (N/p)$. Here it is assumed that N is divisible by p. Finally, denote the Fourier transform of $X_{jk}(m\Delta t)$ by $X_{jk}(f)$. The average crosspower spectral estimate between channels i and j is then given by

$$\Phi_{ij}(f) = \frac{1}{p} \sum_{k=1}^{p} X_{ik}(f)\, X^*_{jk}(f)$$

*Haubrich, Richard A., 1965: Earth Noise, 5 to 500 millicycles per second, J.G.R. v. 70, n. 6, 15 Mar., p. 1415-1427.

where $X^{*}_{jk}(f)$ is the complex conjugate of $X_{jk}(f)$. Similarly, the autopower for the $i^{th}$ channel is given by

$$\Phi_{ii}(f) = \frac{1}{p} \sum_{k=1}^{p} X_{ik}(f)\, X^{*}_{jk}(f)$$

These relations are equivalent to methods given by Blackman and Tukey,* and relations are equivalent to forming the correlation function $R^{k}_{ij}(\tau)$ for the $k^{th}$ segment where $R^{k}_{ij}(\tau)$ is given by

$$R^{k}_{ij}(\tau) = \frac{1}{M} \sum_{m=1}^{M-|\tau|} X_{ik}(m\Delta t)\, X_{jk}(m\Delta t + |\tau|)$$

and then taking the Fourier transform of the product $h(\tau)\, R^{k}_{ij}(\tau)$, where $h(\tau)$ is a time window. For this process, $h(\tau)$ is given by

$$\begin{aligned} h(\tau) &= 1 - \frac{\tau}{M}, & \tau &= 0, \pm 1, \ldots, \pm M \\ &= 0, & \tau &= \pm(M+1), \pm(M+2), \ldots \end{aligned}$$

Note that for each $R^{k}_{ij}(\tau)$ formed, a spectral estimate $\Phi^{k}_{ij}(f)$ is obtained. The spectral estimates are then averaged, thereby increasing the reliability and stability of the spectral estimate. The final estimate $\Phi_{ij}(f)$ is equivalent for both processes. The full value of the FFT with Haubrich's method is easily seen in terms of increased efficiency over the Blackman and Tukey method.

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*Blackman, R. B., and J. W. Tukey, 1958: The Measurement of Power Spectra, Dover Publications, Inc., New York.

## C. SOLUTION OF THE MATRIX EQUATION

In Section II it was shown that the Wiener filters are obtained by solving a matrix equation of the form AX = F, where A is a square Hermitian matrix and X and F are both column vectors. The components of the column vector X are the filter weights; the column vector F is the signal model, and the matrix A is the crosspower matrix computed using the techniques previously discussed. All that remains is to solve for X in terms of the known quantities A and F. The method currently used and computationally more efficient than any other technique used to date is the Square-Root Method. This method is approximately four times faster than the Gauss' Method, which was the last technique used.

Since A is a square, Hermitian matrix, it can be expressed as the product of two triangular matrices of which one is the transpose of the other. Thus, let A be given by

$$A = S'S$$

where

$$S = \begin{bmatrix} S_{11} & S_{12} \dots & S_{1n} \\ 0 & S_{22} \dots & S_{2n} \\ 0 & 0 \quad \dots & S_{nn} \end{bmatrix}$$

Forming the product S′S gives the elements of A in terms of the elements of S. The elements of A are given by

$$a_{ij} = S_{1i}\, S_{1j} + S_{2i}\, S_{2j} + \dots + S_{ii}\, S_{ij}, \qquad i < j$$

$$a_{ii} = S_{1i}^{2} + S_{+i}^{+} + \dots + S_{ii}^{2}, \qquad i = j$$

Solving the preceding equations for the $S_{ij}$ gives

$$S_{1i} = \sqrt{a_{11}} \quad S_{1j} = \frac{a_{ij}}{S_{11}}$$

$$S_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} S_{ki}^2}\,, \qquad i > 1$$

$$S_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} S_{ki} S_{kj}}{S_{ii}}\,, \qquad j > i$$

$$S_{ij} = 0, \qquad i > j$$

By expressing A as $S'S$, solving the equation $AX = F$ now becomes equivalent to solving the two equations

$$S'K = F, \qquad SX = K$$

The elements of K are determined by recurrence relations similar to those given above. They are

$$k_1 = \frac{f_1}{S_{11}}$$

$$k_i = \frac{f_i - \sum_{\ell=1}^{i-1} S_{\ell i} k_\ell}{S_{ii}}\,, \qquad i > \ell$$

The final solution is found from the relation

$$X_n = \frac{k_n}{S_{nn}}$$

$$X_i = \frac{k_i - \sum_{\ell=i+1}^{n} S_{i\ell}\, S_\ell}{S_{ii}}, \qquad i < n$$

The advantage of the above technique is that it greatly reduces the number of calculations required to find the $X_i$.

This section outlined the computational procedures for implementing the design of Wiener filters on a general purpose computer. Many factors influence how closely the filters designed by the above techniques approximate the theoretical filters. However, by carefully applying the digital techniques, the approximation can be made very close.

SECTION IV

# FREQUENCY WAVENUMBER SPECTRA

A particularly useful tool in the study of Wiener filter performance is the frequency-wavenumber spectrum. The frequency-wavenumber spectrum provides qualitative information about the noise field. This technique provides a method for determining the relative distribution of energy as a function of both frequency and wavenumber, and the wavenumber parameter is directly relatable to spatial parameters. Thus, the energy distribution of the noise field can be represented in terms of space and frequency. Section IV presents a theoretical discussion of the frequency-wavenumber spectrum and a description of how the spectrum can be computed; other computational techniques will not be discussed.

## A. FREQUENCY-WAVENUMBER POWER-DENSITY SPECTRUM

Consider a plane wave of wavelength $\lambda$ propagating in the direction specified by the direction cosines $(\gamma_1 \ldots, \gamma_n)$ in an n-dimensional space. Let its time waveform at the origin by given by

$$g(t) = \cos 2\pi(f_0 t + c)$$

and let $\underline{k}_0$ be the vector wavenumber $1/\lambda(\gamma_1 \ldots, \gamma_n)$, which points in the direction of propagation of the plane wave and has the magnitude $1/\lambda$.

Since the magnitude of the velocity of propagation $\underline{V}_p$ is equal to the frequency times the wavelength $\lambda$ and since the perpendicular distance to any point $\underline{x}$ from a plane through the origin with a normal specified by the direction cosines $(\gamma_1 \ldots, \gamma_n)$ is given by the formula

$$(\gamma_1 \ldots, \gamma_n) \cdot \underline{x}$$

the plane wave will arrive at the position of the point $\underline{x}$ after a time lag

$$\tau = \frac{(\gamma_1, \ldots, \gamma_n) \cdot \underline{x}}{f_0 \lambda} = \frac{\underline{k}_0 \cdot \underline{x}}{f_0}$$

Therefore, the equation of this plane wave at position $\underline{x}$ is given by

$$g(\underline{x}, t) = \cos 2\pi(f_0 t + c - \underline{k}_0 \cdot \underline{x})$$

If $\underline{y}$ is the vector displacement between the two positions $\underline{x} + \underline{y}$ and $\underline{x}$, the space-time correlation function $\varphi(\underline{y}, \tau)$ is the average of the product of the two functions $g(\underline{x}, t)$ and $g(\underline{x} + \underline{y}, t + \tau)$ over all values $\underline{x}$ of position and all values $\tau$ of time:

$$\varphi(\underline{y}, \tau) = \lim_{\substack{T \to \infty \\ L \to \infty}} \frac{1}{2^{n+1} T L^n} \int_{-T}^{T} \underbrace{\int_{-L}^{L} \cdots \int_{-L}^{L}}_{n} g(\underline{x}, t)\, g(\underline{x} + \underline{y}, t + \tau)\, d\underline{x}\, dt$$

where $d\underline{x}$ is vector shorthand for $dx_1 \text{----} dx_n$.

Since

$$g(\underline{x}, \tau) = \cos 2\pi (f_0 t + c - \underline{k}_0 \cdot \underline{x})$$

$$= \mathrm{Re}\left[e^{i2\pi(f_0 t + c - \underline{k}_0 \cdot \underline{x})}\right]$$

and

$$g(\underline{x} + \underline{y}, t + \tau) = \cos 2\pi (f_0 t + f_0 \tau + c - \underline{k}_0 \cdot \underline{x} - \underline{k}_0 \cdot \underline{y})$$

$$= \mathrm{Re}\left[e^{i2\pi(f_0 t + f_0 \tau + c - \underline{k}_0 \cdot \underline{x} - \underline{k}_0 \cdot \underline{y})}\right]$$

the space-time correlation function $\varphi(\underline{y}, \tau)$ reduces to

$$\varphi(\underline{y}, \tau) = \tfrac{1}{2} \operatorname{Re}\left[e^{i2\pi(f_0 \tau - \underline{k}_0 \cdot \underline{y})}\right]$$

after (n + 1)-fold integration over the variables $x_1, \ldots, x_n$ and $\tau$.

Let $g_j(t)$ and $g_\ell(t)$ denote the time waveforms of the plane wave at the positions $\underline{x}_j$ and $\underline{x}_\ell$, respectively. Then the time correlation function

$$\varphi_{j\ell}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g_j(t)\, g_\ell(t+\tau)\, dt$$

between the functions $g_j$ and $g_\ell$ at a time lag $\tau$ is also equal to

$$\tfrac{1}{2} \operatorname{Re}\left[e^{i2\pi(f_0\tau - \underline{k}_0 \cdot \underline{y})}\right] = \tfrac{1}{2} \cos 2\pi\,(f_0\tau - \underline{k}_0 \cdot \underline{y})$$

where $\underline{y}$ is the vector displacement $\underline{x}_\ell - \underline{x}_j$. When, as in this instance, the time correlation function $\varphi_{j\ell}(\tau)$ between the time waveforms at two positions $\underline{x}_j$ and $\underline{x}_\ell$ does not depend upon their absolute locations — but only upon their relative locations, together with the delay time $\tau$ — the wavefield is said to be <u>space-stationary</u> (in the strong sense). Any superposition of plane waves is space-stationary. For space-stationary wavefields, the space-time correlation function $\varphi(\underline{y}, \tau)$ is equal to the time correlation function $\varphi_{j\ell}(\tau)$ between any two points $\underline{x}_j$ and $\underline{x}_\ell$ such that $\underline{y} = \underline{x}_\ell - \underline{x}_j$. This fundamental point will be a key building block in a later discussion.

The space-time to frequency-wavenumber Fourier transform $\Phi(f, \underline{k})$ of the space-time crosscorrelation function $\varphi(\underline{y}, \tau)$, called the frequency-wavenumber power density spectrum, is defined to be

$$\Phi(f, \underline{k}) = \lim_{\substack{T \to \infty \\ L \to \infty}} \int_{-T}^{T} \underbrace{\int_{-L}^{L} --- \int_{-L}^{L}}_{n} \varphi(\underline{y}, \tau) \, e^{-i2\pi(f\tau - \underline{k} \cdot \underline{y})} \, d\underline{y} d\tau$$

where $d\underline{y}$ is vector shorthand for $dy_1 --- dy_n$. If $\varphi(\underline{y}, \tau) = \frac{1}{2} \cos 2\pi(\underline{k}_0 \cdot \underline{y} - f_0 \tau)$,

$$\Phi(f, \underline{k}) = \frac{\delta(f - f_0, \underline{k} - \underline{k}_0)}{2} + \frac{\delta(f + f_0, \underline{k} + \underline{k}_0)}{2}$$

where $\delta$ is the Dirac delta function.

Figure 1 shows the frequency-wavenumber power density spectrum of a plane wave propagating in a 1-dimensional space.

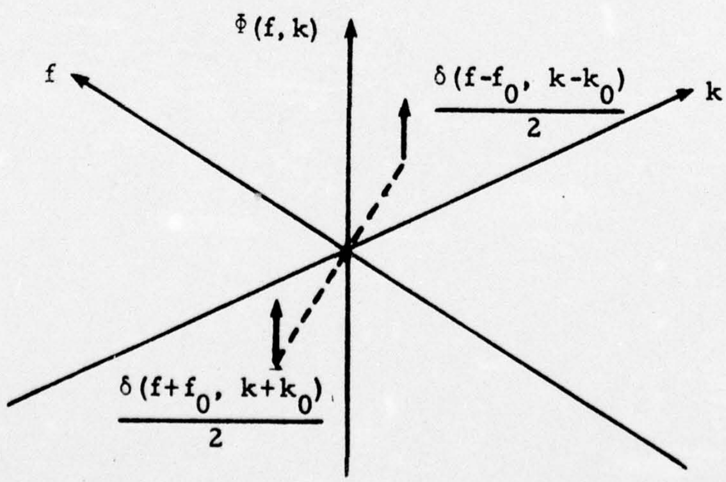


Figure 1. Frequency-Wavenumber Power Density Spectrum of Plane Wave Given by Equation $g(x, t) = \cos 2\pi (f_0 t + c - k_0 x)$

This result is an extension to n + 1 dimensions of the frequency power density spectrum

$$\Phi(f) = \frac{\delta(f-f_0)}{2} + \frac{\delta(f+f_0)}{2}$$

for the sinusoidal time function $g(t) = \cos 2\pi(f_0 t + c)$.

## B. SPECTRAL WINDOW

When frequency-wavenumber spectra are computed using the outputs $g_j(t)$ of a set of array elements specified by the position vectors $\{\underline{x}_j \mid j = 1, 2, \ldots, m\}$, it is only possible to obtain an estimate of the true frequency-wavenumber spectrum $\Phi(f, \underline{k})$. Instead of computing the integral

$$\Phi(f, \underline{k}) = \lim_{\substack{T\to\infty \\ L\to\infty}} \int_{-T}^{T} \underbrace{\int_{-L}^{L} \cdots \int_{-L}^{L}}_{n} \varphi(\underline{y}, \tau)\, e^{-i2\pi(f\tau - \underline{k}\cdot\underline{y})} d\underline{y}\, d\tau$$

it is only possible to compute a summation over the vector displacements $\underline{y}$ corresponding to the various sensor pairs of a spatial array. The set of unique vector displacements between all possible combinations of sensor pairs is called the set of points in correlation space corresponding to the array. If several sensor pairs lie at the same relative vector displacements, they are said to be redundant.

If a wavefield is composed of a superposition of plane waves, it is space-stationary in the strong sense; therefore, the space-time correlation function $\varphi(\underline{y}, \tau)$ may be obtained by computing the time correlation function

$$\varphi_{j\ell}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g_j(t) g_\ell(t+\tau)\, dt$$

between the sensor outputs $g_j(t)$ and $g_\ell(t)$ of two sensors located at $\underline{x}_j$ and $\underline{x}_\ell$, respectively, provided that the vector displacement $\underline{x}_\ell - \underline{x}_j$ is equal to $\underline{y}$. This fact permits a spatial sampling of the space-time correlation function $\varphi(\underline{y}, \tau)$ over the set of points in correlation space corresponding to an array.

If the sensor outputs of an array are not approximately consistent with a space-stationary wavefield, the frequency-wavenumber spectrum is not an appropriate analysis technique. The sensor outputs are then best characterized in terms of the auto- and crosspower spectra $\Phi_{j\ell}(f)$ between sensors.

A common method of computing frequency-wavenumber spectra is to compute the power spectra

$$\Phi_{j\ell}(f) = \int_{-\infty}^{\infty} \varphi(\underline{x}_\ell - \underline{x}_j, \tau)\, e^{-i2\pi f\tau}\, d\tau$$

and then the summation

$$\sum_j \sum_\ell \Phi_{j\ell}(f) e^{-i2\pi \underline{k} \cdot (\underline{x}_j - \underline{x}_\ell)}$$

to complete the space-time to frequency-wavenumber Fourier transform. (If several correlation space vectors $\underline{x}_j - \underline{x}_\ell$ are redundant, their averaged power spectrum is used only once in this summation.)

This summation is equivalent to multiplying the Fourier transform integrand by a sum of spatial Dirac delta functions

$$\sum_m \delta(\underline{y} - \underline{y}_m)$$

where the $\underline{y}_m$ are the various vector displacements comprising the correlation space corresponding to the array used:

$$\lim_{\substack{T\to\infty \\ L\to\infty}} \int_{-T}^{T} \underbrace{\int_{-L}^{L} \cdots \int_{-L}^{L}}_{n} \varphi(\underline{y}, \tau)\, e^{-i2\pi(f\tau - \underline{k}\cdot\underline{y})} \left[\sum_m \delta(\underline{y} - \underline{y}_m)\right] d\underline{y}\, d\tau$$

$$= \sum_m \Phi_{\underline{y}_m}(f)\, e^{i2\pi\underline{k}\cdot\underline{y}_m} = \sum_j \sum_\ell \Phi_{\ell j}(f)\, e^{i2\pi\underline{k}\cdot(\underline{x}_j - \underline{x}_\ell)}$$

$$= \sum_j \sum_\ell \Phi_{\ell j}(f)\, e^{-i2\pi\underline{k}\cdot(\underline{x}_\ell - \underline{x}_j)}$$

Here

$$\Phi_{\underline{y}_m}(f) = \int_{-\infty}^{\infty} \varphi(\underline{y}_m, \tau)\, e^{-i2\pi f\tau} d\tau = \Phi_{\ell j}(f)$$

where

$$\varphi(\underline{y}_m, \tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(\underline{x}_\ell, t)\, g(\underline{x}_j, t+\tau)\, dt$$

because we let

$$\underline{y}_m = \underline{x}_j - \underline{x}_\ell$$

A well-known result from the theory of operational calculus known as the convolution theorem states that the complex Fourier transform of the product of two functions of a real variable, one of which may be a generalized function such as the Dirac delta function, is the convolution of the individual Fourier transforms of each of the functions:

$$\int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-i2\pi yx} dx = F_f(y) * F_g(y)$$

$$= \int_{-\infty}^{\infty} F_f(z)\, F_g(y - z)\, dz$$

$$= \int_{-\infty}^{\infty} F_f(y - z)\, F_g(z)\, dz$$

where $F_f(y)$ is the Fourier transform

$$\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi yx} dx$$

and $F_g(y)$ is the Fourier transform

$$\int_{-\infty}^{\infty} g(x)\, e^{-i2\pi yx} dx$$

The n-fold application of this result implies that multiplication by the quantity $\sum_m \delta(\underline{y} - \underline{y}_m)$ corresponds to convolution of the true frequency-wavenumber power spectrum $\Phi(f, \underline{k})$ with the Fourier transform of $\sum_m \delta(\underline{y} - \underline{y}_m)$, which is equal to $\sum_m e^{i2\pi \underline{k} \cdot \underline{y}_m}$. The quantity $W(\underline{k}) = \sum_m e^{i2\pi \underline{k} \cdot \underline{y}}$ is known as the <u>spectral window</u> of the array used in computing the spectrum $\Phi(f, \underline{k})$. Thus, the frequency-wavenumber spectrum obtained will be

$$\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} \Phi(f, \underline{k} - \underline{\kappa})\, W(\underline{\kappa})\, d\underline{\kappa} = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} \Phi(f, \underline{\kappa})\, W(\underline{k} - \underline{\kappa})\, d\underline{\kappa}$$

instead of the true frequency-wavenumber spectrum $\Phi(f, \underline{k})$. If, at a particular frequency, all of the true power is concentrated at the wavenumber $\underline{k}_0$, the frequency-wavenumber power density spectrum obtained will be a constant times the wavenumber spectral window W $(\underline{k})$ shifted by $\underline{k}_0$ in wavenumber space.

Since the spectral window is the sum of a finite number of complex sinusoids $e^{i2\pi \underline{k} \cdot \underline{y}_m}$, it cannot be the spike $\delta(\underline{k})$ required to eliminate distortion of the true frequency-wavenumber spectrum. In fact, to achieve the ideal spectral window, i. e., the Dirac delta function $\delta(\underline{k})$, the correlation space sampling function must be a function equal to 1 at all points of the n-dimensional correlation space $(y_1, \ldots, y_n)$. The spectral window of any array of discrete-point sensors will have rounded peaks over all of wavenumber space. The peak at $\underline{k} = \underline{0}$ is called the main lobe. All other peaks are called sidelobes. To minimize the distortion of frequency-wavenumber spectra, the spectral window sidelobes should be as low as possible.

The same method can be applied to determine the angle of incidence for waves propagating across a 2-dimensional array. For a particular frequency, it will be possible to draw rings of constant apparent velocity or (equivalently) of constant angle of incidence. The angle of incidence $\theta$ will be determined similarly to the 1-dimensional case by the formula

$$\theta = \sin^{-1}\left( |\underline{k}|\ |\underline{V}_p|/f \right) = \sin^{-1}\left( |\underline{V}_p|/|\underline{V}_a| \right)$$

Each ring will be centered at the origin of the wavenumber spectrum for a particular frequency and will be the circle determined by taking a horizontal slice at the frequency f of the corresponding constant-velocity cone in the 3-dimensional representation of the frequency-wavenumber spectrum.

## C. COMPUTATION OF FREQUENCY-WAVENUMBER SPECTRA

The computation of a frequency-wavenumber power-density spectrum $\Phi(f, \underline{k})$ is similar to the computation of a frequency power-density spectrum $\Phi(f)$. In the case of a frequency power-density spectrum, a Fourier transform from time $\tau$ to frequency f is made while, in the case of a frequency-wavenumber spectrum, a Fourier transform from space and time $(\underline{y}, \tau)$ to frequency and wavenumber $(f, \underline{k})$ is performed. In the conventional power-density spectrum computation, the autocorrelation $\varphi(\tau)$ is transformed to yield the power density spectrum

$$\Phi(f) = \int_{-\infty}^{\infty} \varphi(\tau)\, e^{-i2\pi f\tau}\, d\tau$$

On the other hand, in the frequency-wavenumber power density spectrum computation, the space-time correlation function $\varphi(\underline{y}, \tau)$ is transformed to yield the frequency-wavenumber spectrum

$$\Phi(f, \underline{k}) = \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} \varphi(\underline{y}, \tau)\, e^{-i2\pi(f\tau - \underline{k} \cdot \underline{y})}\, d\underline{y}\, d\tau$$

Here, as mentioned earlier, n indicates an n-fold integration corresponding to the number of dimensions of the array, and $d\underline{y}$ is vector shorthand for $dy_1 \text{ --- } dy_n$.

Earlier, it was shown that this definition of a frequency-wavenumber power density spectrum does indeed yield a physically meaningful representation of the space-time wavefield. There, a frequency-wavenumber power-density spectrum $\Phi(f, \underline{k})$ was shown to transform a unit-amplitude plane

wave of frequency f and wavenumber $\underline{k}$ into two half-strength impulses (i.e., Dirac delta functions) at $(f, \underline{k})$ and $(-f, -\underline{k})$ just as a frequency power-density spectrum transforms a unit-amplitude sinusoidal waveform of frequency f into two half-strength impulses at f and -f.

The algorithm for the transformation of a space-time correlation function $\varphi(\underline{v}, \tau)$ to the corresponding frequency-wavenumber power density spectrum $\Phi(f, \underline{k})$ consists of two steps.

The first step is the computation of the crosspower statistics of the wave energy in the form of interchannel crosspower functions between the outputs of the elements in the sensing array. The techniques for computing the crosspower spectra is the same as that discussed in Sec. III.

The second step is the summation

$$\sum_{j} \sum_{\ell} \Phi_{j\ell}(f) \; e^{-i2\pi \underline{k} \cdot (\underline{x}_j - \underline{x}_\ell)}$$

which completes the transformation.

Provided that energy travels unattenuated across the array in the form of plane waves with negligible refraction, reflection or diffraction, the crosspower spectra $\Phi_{jl}(f)$ corresponding to a particular vector displacement will be identical. This assumption of strong space-stationarity will be approximately correct if energy sources are distant from the array and the medium in the vicinity of the array is near-uniform. The averaging of all transforms $\Phi_{jl}(f)$ corresponding to a particular vector displacement will tend to reduce the effects of non-space stationarity.

## SECTION V
## ARRAY GAIN

In the investigation of Wiener multichannel filtering, it is necessary to have some basis for comparison between multichannel filter performance and the performance of a well-understood standard. The basis chosen is the amount of array gain achieved compared to the gain of a time-shift-and-sum beamformer. Time-shift-and-sum beamforming is the type of beamforming currently used in a great many acoustic systems and is well understood by a great many people. Therefore, comparing the amounts of array gain obtained by Wiener filtering to that for time-shift-and-sum beamforming is both realistic and meaningful.

The array gain of a particular processing system is defined to be the output signal-to-noise ratio divided by the input signal-to-noise ratio of a single sensor. For multichannel filtering or time-shift-and-sum beamforming, the output signal-to-noise ratio for a given frequency is given by the following matrix equation:

$$(S/N)_{out} = \frac{A^H \Phi^S A}{A^H \Phi^N A}$$

where $\Phi^S$ is the signal crosspower spectrum matrix, $\Phi^N$ is the noise crosspower spectrum matrix, A is a column matrix whose components are either the optimum filter set or the time-shift-and-sum filter set, and H denotes the transpose conjugate. If the matrix A consists of the time-shift-and sum filter set, then A is given by

$$A = \begin{bmatrix} e^{i2\pi f \tau_1} \\ e^{i2\pi f \tau_2} \\ \cdot \\ \cdot \\ \cdot \\ e^{i2\pi f \tau_n} \end{bmatrix}$$

where each $\tau_i$i,i = 1, 2, ..., n, is the time delay required to align an incoming plane wave in phase so that coherent addition will occur when the outputs of all the sensors are summed.

There is a problem when it comes to defining the input signal-to-noise ratio. If one sensor in the array is chosen from which the input signal-to-noise ratio is determined, then the assumption has been made that the noise level is the same at all other sensors. This assumption is hard to validate. For this reason, the input signal-to-noise ratio is taken to be

$$(S/N)_{in} = \frac{\Phi^S(f)}{\frac{1}{n} \sum_{j=1}^{n} \Phi_j^N(f)}$$

where $\Phi^S(f)$ is the signal autopower spectrum and $\Phi_j^N(f)$ is the noise autopower spectrum as measured by the $j^{th}$ sensor.

Using the above definitions for input and output signal-to-noise ratios gives the array gain measurement G(f)

$$G(f) = \frac{(S/N)_{out}}{(S/N)_{in}}$$

$$= \frac{A^H(f)\, \Phi^S(\iota)\, A(f)}{A^H(f)\, \Phi^N(f)\, A(f)} \Bigg/ \frac{\Phi^S(f)}{\frac{1}{n} \sum_{j=1}^{n} \Phi_j^N(f)}$$

Let A be the matrix of optimum filter weights and let B be the matrix of time-shift-and-sum filter weights. To compare the performance of multichannel filtering with that of time-shift-and sum beamforming, the

ratio of the array gain achieved by optimum beamforming to that achieved by time-shift-and-sum beamforming is formed. This ratio, denoted by C(f), is given by

$$C(f) = \frac{A^H \Phi^S A}{A^H \Phi^N A} \Bigg/ \frac{B^H \Phi^S B}{B^H \Phi^N B}$$

The function C(f) can be used to compare the performance of the two systems when trying to process against the same noise field. The main advantage achieved by taking the ratio of the array gains is that C(f) is not dependent on the input signal-to-noise ratio. Thus, the outputs of the processor can be directly related.